

# DULL CUT OFF FOR CIRCULANTS

**ABSTRACT.** Families of symmetric simple random walks on Cayley graphs of Abelian groups with a bound on the number of generators are shown to never have sharp cut off in the sense of [1], [3] or [5] for the convergence to the stationary distribution in the total variation norm. This is a situation of bounded degree and no expansion. Sharp cut off or the cut off phenomenon has been shown to occur in families such as random walks on a hypercube [1] in which the degree is unbounded as well as on a random regular graph where the degree is fixed, but there is expansion [4]. These examples agree with Peres' conjecture in [3] relating sharp cut off, spectral gap and mixing time.

## 1. INTRODUCTION

A random walk on a finite Abelian group  $G$  is called type  $r$  if  $|G| \geq \pi^r$  and the walk is given by applying one of the elements  $\{\pm a_i, 0\}_{i \in [r]}$  each with equal probability or equivalently a simple random walk on the Cayley graph for the generating set  $\{\pm a_i\}$  with the probability of staying at a vertex the same as that of following each edge.

**Theorem 1.** *No family of walks all of the same type has sharp cut off.*

See Section eight of [5] for more on the convergence rate of such walks.

If  $A$  is an irreducible symmetric Markov matrix with unique stationary distribution  $\mathbf{v}_0$  (so that  $A\mathbf{v}_0 = \mathbf{v}_0$  and  $|\mathbf{v}_0|_1 = 1$ ) and  $\mathbf{x}_0 = (1, 0, \dots, 0)$  write

$$d_A(t) = |A^t \mathbf{x}_0 - \mathbf{v}_0|_1,$$

$$t_A(d) = \max\{t | d_A(t) \geq d\},$$

$\{\lambda_k\} \subseteq (-1, 1]$  for the eigenvalues of  $A$  with  $\lambda_0 = 1$  and  $|\lambda_m| = \max_{k \neq 0} |\lambda_k| < 1$ .

**Definition 2.** *A family  $\{A_i\}$  of irreducible symmetric Markov matrices has sharp cut off if*

$$\lim_{n \rightarrow \infty} \frac{t_{A_n}(\epsilon)}{t_{A_n}(1 - \epsilon)} = 1$$

for every  $\epsilon \in (0, \frac{1}{2})$ .

See Definition 3.3 in [5].

## 2. PROOF OF THEOREM 1

The proof is given for the case in which every group is cyclic. Other Abelian groups work similarly.

*Proof.* Consider a type  $r$  walk on  $\mathbb{Z}/n\mathbb{Z}$  with step directions  $\{\pm a_i\}$  and irreducible symmetric Markov transition matrix  $A$ . Note the Fourier expansion:

$$\lambda_k = \sum_{i=1}^r \frac{2}{2r+1} \cos\left(2\pi \frac{ka_i}{n}\right) + \frac{1}{2r+1}.$$

**Lemma 3.**

$$\lambda_m^{2t} \leq d_A^2(t) \leq \sum_{k \neq 0} \lambda_k^{2t}.$$

*Proof.* For the left inequality note that  $A = A^*$  is self adjoint and the stationary distribution is  $\mathbf{v}_0 = \frac{1}{n}\mathbf{1}$  and write  $\mathbf{v}_m$  for the eigenvector with  $A\mathbf{v}_m = \lambda_m\mathbf{v}_m$  and  $|\mathbf{v}_m|_1 = 1$ . Since  $A$  is a transition matrix for a random walk on a Cayley graph for  $G$  it commutes with rotation (action by  $G$ ). Thus  $\mathbf{v}_m$  is an eigenvector for rotation and hence all entries of  $\mathbf{v}_m$  have the same norm, which by the normalization is  $|\mathbf{v}_m|_\infty = \frac{1}{n}$ . Since  $d_A(t) = \max_{\mathbf{v}} \frac{\langle A^t \mathbf{x}_0 - \mathbf{v}_0, \mathbf{v} \rangle}{|\mathbf{v}|_\infty}$  this gives  $d_A(t) \geq n|\langle A^t \mathbf{x}_0 - \mathbf{v}_0, \mathbf{v}_m \rangle| = |\lambda_m|^t$ .

For the right inequality if  $\mathbf{v}_k$  is the eigenvector of  $A$  with eigenvalue  $\lambda_k$  and every entry having norm  $\frac{1}{n}$  then  $|\langle \mathbf{x}_0, \mathbf{v}_k \rangle| = \frac{1}{n}$  so that  $d_A^2(t) = |A^t \mathbf{x}_0 - \mathbf{v}_0|_1^2 \leq n|A^t \mathbf{x}_0 - \mathbf{v}_0|_2^2 = \sum_{k \neq 0} \lambda_k^{2t}$ .  $\square$

Replace trigonometric functions with exponentials using that if  $x \in [-\frac{3}{2}\pi, \frac{3}{2}\pi]$  then  $\cos(x) \leq e^{-\frac{1}{2\pi^2}x^2}$  and if  $x \in [-1, 1]$  then  $e^{-x^2} \leq \cos(x)$ .

Commute sums with exponentials using that concavity of  $e^x$  and  $e^{-x^2}$  so that  $e^{-\frac{1}{2}a^2 - \frac{1}{2}b^2} \leq \frac{1}{2}e^{-a^2} + \frac{1}{2}e^{-b^2}$  for any  $a$  and  $b$  and if  $a, b \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$  then  $\frac{1}{2}e^{-a^2} + \frac{1}{2}e^{-b^2} \leq e^{-(\frac{1}{2}a + \frac{1}{2}b)^2}$ .

Eliminate the one norm by using that  $|v|_1 \geq |v|_2$  and write  $\langle x \rangle \in (-\frac{1}{2}, \frac{1}{2}]$  for the smallest translate of  $x$  by an integer.

Combining these gives for every  $k$  that

$$\lambda_k \leq e^{-\frac{8}{(2r+1)^2}|\langle \frac{ka}{n} \rangle|_2^2}.$$

Since  $n \geq \pi^r$  and  $\text{Vol}[\mathbb{R}^n/(\mathbb{Z}^n + \frac{a}{n})] = \frac{1}{n}$  there is some  $k$  with every  $|\langle \frac{ka}{n} \rangle| \leq \frac{1}{2\pi}$  so that

$$|\lambda_m| \geq |\lambda_k| \geq e^{-\frac{8\pi^2}{2r+1}|\langle \frac{ka}{n} \rangle|_2^2}.$$

**Lemma 4.** For every  $r$  there is  $\kappa_r$  so that if  $\Lambda$  is a full rank lattice in  $\mathbb{R}^r$  and  $\mu = \min_{\mathbf{0} \neq \sigma \in \Lambda} \{|\sigma|_2\}$  then

$$\sum_{\sigma \in \Lambda} e^{-a|\sigma|_2^2} \leq 1 + \frac{\kappa_r}{e^{a\mu^2} - 1}$$

or

$$\sum_{\sigma \in \Lambda} e^{-a|\sigma|_2^2} \leq 1 + \frac{\kappa_r}{(e^{a\mu^2} - 1)^r}.$$

*Proof.*

**Definition 5.** A full rank lattice  $\Lambda \subseteq \mathbb{R}^r$  is tight if  $\mu = \min_{\mathbf{0} \neq \sigma \in \Lambda} \{|\sigma|_2\} = 1$  and there is a basis for  $\Lambda$  so that  $|\{\sigma \in \Lambda \mid |\sigma|_2 = 1\}|$  is maximal among lattices generated by rescalings of this basis with shortest nonzero element of length one.

Note that the tight lattices form a compact subset of all full rank lattices in  $\mathbb{R}^n$ .

**Definition 6.** An acute cone in a lattice  $\Lambda \subseteq \mathbb{R}^n$  is a free submonoid generating  $\Lambda$  as a group and with all inner products of nonzero elements positive.

Note that acuteness of a cone is an open condition in the space of lattices and every lattice has a finite decomposition into acute cones. Write

$$\kappa_r = \max_{\Lambda \subseteq \mathbb{R}^n} \min_{\Lambda = \cup_{i \in I} C_i} |I|$$

where  $\Lambda$  is a full rank lattice and  $\{C_i\}_{i \in I}$  is a decomposition into interiors of acute cones. By compactness  $\kappa_r$  is finite. Note that for any rank  $r$  lattice  $\Lambda$  with  $\mu = \min_{\sigma \in \Lambda} \{|\sigma|_2\} = 1$  there is a tight lattice  $\Lambda'$  with  $\sum_{\sigma \in \Lambda} e^{-a|\sigma|_2^2} \leq \sum_{\sigma' \in \Lambda'} e^{-a|\sigma'|_2^2}$  obtained by rescaling some generators.

If  $\Omega$  is the interior of an acute cone of dimension  $d > 0$  in a tight lattice then

$$\sum_{\omega \in \Omega} e^{-a|\omega|_2^2} \leq \sum_{\omega \in \Omega} e^{-a|\omega|_2^2} \leq \sum_{\omega \in \mathbb{Z}_{>0}^d} e^{-a|\omega|_2^2} = (e^a - 1)^{-d} \leq \max\{(e^a - 1)^{-1}, (e^a - 1)^{-r}\}.$$

□

Thus using the second inequality from Lemma 3

$$\begin{aligned} d_A^2(t) &\leq \sum_{\sigma \in \mathbb{Z}^r + \frac{a}{n}} e^{-\frac{16t}{(2r+1)^2} |\sigma|_2^2} - 1 \leq \max \left\{ \frac{\kappa_r}{e^{\frac{16t}{(2r+1)^2} |\sigma_m|_2^2} - 1}, \frac{\kappa_r}{\left(e^{\frac{16t}{(2r+1)^2} |\sigma_m|_2^2} - 1\right)^r} \right\} \\ &\leq \max \left\{ \frac{\kappa_r}{\lambda_m^{\frac{-2t}{\pi^2(2r+1)}} - 1}, \frac{\kappa_r}{\left(\lambda_m^{\frac{-2t}{\pi^2(2r+1)}} - 1\right)^r} \right\}. \end{aligned}$$

Using the first inequality from Lemma 3 at  $d = \epsilon$ , the above inequality at  $d = 1 - \epsilon$  and taking  $\epsilon < e^{-2\pi^2(2r+1)\kappa_r}$  gives

$$\frac{t_A(\epsilon)}{t_A(1 - \epsilon)} \geq \frac{-\ln \epsilon}{2\pi^2(2r+1)\kappa_r} > 1.$$

□

Note that the same bound as above gives  $t_A(\frac{1}{2}) \leq \frac{\kappa_r 2\pi^2(2r+1)}{-\ln |\lambda_m|}$  so that  $(1 - |\lambda_m|)t_A(\frac{1}{2})$  is uniformly bounded and this class of examples agrees with Peres' conjecture in [3].

## REFERENCES

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